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COMMENT

Multifractal analysis of chaotic power spectra

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Abstract. We analyse the scaling structure of power spectra arising from chaotic dynamical systems. The observation of anomalous scaling in spectral parameters can be understood by the use of multifractal analysis in the frequency domain. This analysis provides numerical tools for evaluating different chaotic behaviour. The frequency behaviour of oscillatory chaos seems to suggest the hypothesis of phase transition in the $f(\alpha)$-spectrum.

Spectral analysis plays an important role in the study and characterization of complex dynamical systems. The main interest in working with power spectra is because in many cases only a one-dimensional time series of a given dynamics is known. The procedures deriving from Takens approach [1], for reconstructing dynamics are limited by the algorithmic implementation of box-counting techniques [2]. Alternative procedures must therefore be sought. Though frequency-domain analysis could provide an alternative starting point in the characterization of dynamical complexity so far this approach achieved no promising results. The spectral parameters chosen to characterize chaotic power spectra, namely spectral dimension [3, 4], spectral entropy [5], spectral degree of freedom [6], are almost all subject to criticism [7]. Indeed as noted by Vidal and Lafon [7], the fractal nature of chaotic power spectra is still an open question.

In the generally adopted paradigm in frequency-domain analysis, the basic assumption is that the power spectrum of a chaotic signal has a continuous frequency support [8].

Indeed, if δf is the frequency resolution in evaluting the spectrum using fast fourier transform (FFT), and if $P_i(\delta f)$ are the spectral components of the normalized discrete power spectrum, the discrete spectral entropy $H(\delta f)$

$$H(\delta f) = -\sum_i P_i(\delta f) \log[P_i(\delta f)]$$

is related to the spectral entropy S of the normalized continuous spectral power density $p(f)$

$$S = -\int p(f) \log[p(f)] \, df$$

via the relation discussed by Powell and Percival [5]:

$$H(\delta f) = S - \log(\delta f) \qquad \delta f \to 0. \tag{1}$$

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Of course, this equation is valid only if the integral S exists in the Lebesgue sense [9]. Moreover, under the assumption of integrability, the spectral degree of freedom $F(\delta f)$ of a signal possessing a continuous frequency support admits the expression:

$$F(\delta f) = \log\left[\frac{\delta f}{\Sigma P_i^2(\phi)}\right] = F_0 = \text{constant} \qquad \delta f \to 0. \tag{2}$$

According to (1), (2) the scaling behaviour of $H(\delta f)$ and $F(\delta f)$ with respect to the frequency resolution δf is trivial and the basic parameters are S and F_0 as discussed by Lafon and Vidal in [7].

$H(\delta f)$ and $F(\delta f)$ were evaluated in the case of two well known chaotic systems [10], namely the Lorenz system:

$$\dot{x} = -\sigma(x-y) \qquad \dot{y} = -xz + rx - y \qquad \dot{z} = xy - bz$$

and the Rossler system:

$$\dot{x} = -(y+z) \qquad \dot{y} = x + 0.2z \qquad \dot{z} = 0.2 + xz - Cz.$$

The chosen parameter values are $\sigma = 10$, $r = 28$, $b = 8/3$, $C = 5.7$ which give rise to chaotic behaviour.

The computations were performed using an FFT-algorithm with 2^{16} temporal samples (at the maximal frequency resolution) of the components of the dynamics averaged over ten time series randomly chosen over 2^{18} samples. The zero-frequency component was also filtered in the time series and used Hanning filtering used to minimize sidelobe overlapping (leakage), [5].

The result of this analysis is that relations (1), (2) are not fulfilled (see figure 1). By analogy with the theory of generalized dimension [11], equations (1) and (2) have to be replaced with the more general scaling laws:

$$\begin{aligned} H(\delta f) &= S' - D(1)\log(\delta f) \\ F(\delta f) &= F_0' + [1 - D(2)]\log(\delta f) \end{aligned} \tag{3}$$

where $D(1)$ and $D(2)$ are respectively the information and the correlation dimension of power spectra. Equations (1) and (2) correspond to the limiting case $D(1) = D(2) = 1$. This experimental result can be interpreted [3] by the use of the Riemann–Lebesgue

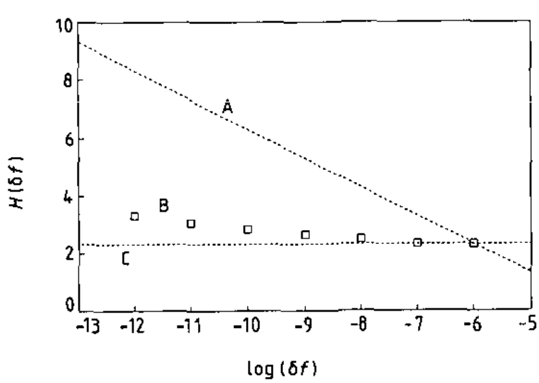


Figure 1. Evidence of anomalous scaling law: spectral entropy scaling for Rossler x-component A. The broken lines represent the spectral entropy scaling for a signal following (1), B, and for a quasi-periodic signal C.

lemma for the Fourier spectrum of a chaotic (non-quasiperiodic) signal. Indeed Blascher and Perdang in [3] assume that the Fourier spectrum of a chaotic signal is characterized by an uncountable set of discontinuities (singularities).

This assumption can be generalized to the statement that to a chaotic power can be associated a spectrum of generalized dimension $D(q)$ [11] related to the scaling of the spectral partition function:

$$\Gamma_q(\delta f) = \sum_i [P_i(\delta f)]^q \sim (\delta f)^{(q-1)D(q)} \qquad q \neq 1 \tag{4}$$

which for $q = 1$ is related to $H(\delta f)$ via the relation:

$$\lim_{q \to 1} \frac{\log \Gamma_q(\delta f)}{(q-1)} = H(\delta f).$$

The values of $D(q)$ can be obtained from the log–log plot of $\Gamma_q(\delta f)$ versus δf.

In (4) it has been assumed that the normalized discrete spectrum could be considered as a probability measure on the frequency support and that the components $P_i(\delta f)$ represent the measure of the box $B_i(\delta f)$ in the δf-covering of the frequency line:

$$P_i(\delta f) = \int_{B_i(\delta f)} \mathrm{d}\mu(f) \tag{5}$$

where $\mu(f)$ is the normalized power spectrum distribution function.

As reported in figures 2 and 3 the assumption of anomalous scaling (4) is confirmed by the experimental data using Lorenz and Rossler systems. Relation (4) and the results of figures 2 and 3 provide a positive a positive answer to the question of the fractal nature of chaotic power spectra. A non-fractal power spectrum is characterized by a constant value of $D(q)$: for periodic and quasi-periodic signals $D(q)$ is uniformly equal to zero; for non-fractal power spectra with a continuous frequency support, $D(q)$ is uniformly equal to one. Fractal power spectra give rise to a non-constant $D(q)$ curve. From (3) and (4) it follows that the frequency behaviour of chaotic dynamics can be described by means of the singularity structure of $\mu(f)$ using the formal machinery of multifractal analysis [12].

The next step was to study the structure of $D(q)$- and $f(\alpha)$-spectra of different components of the same dynamics. Takens' result give rise to the intuitive expectation

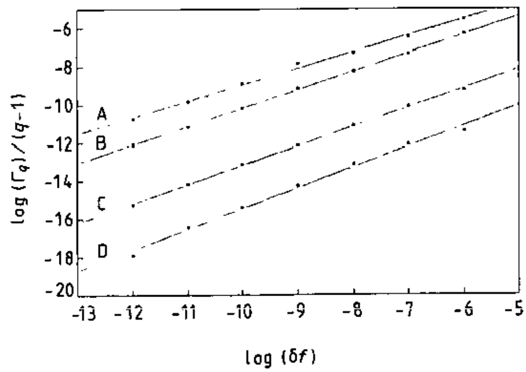


Figure 2. Scaling behaviour of the spectral partition function $\Gamma_q(\delta f)$ for Lorenz x-component: A $q = 10$; B $q = 2$; C $q = -2$; D $q = -10$. The slopes of these lines give the corresponding values of $D(q)$.

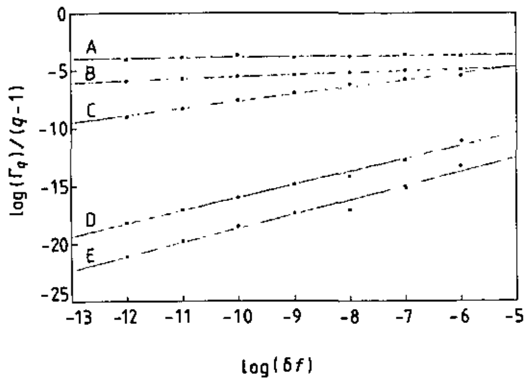


Figure 3. Scaling behaviour of $\Gamma_q(\delta f)$ for Rossler z-component: A $q=10$; B $q=2$; C $q=1$; D $q=-2$; E $q=-5$. For $q=1$ the scaling of $-H(\delta f)$ according to the limit definition of $\Gamma_1(\delta f)$ is shown.

that all the components of a given dynamics admit the same frequency structure even if the power spectra may look different (for example the x-component and z-component of the Lorenz system). Therefore the multifractal scaling structure has to be equal or at least very similar.

In working with power spectra of the same or different dynamics a general criterion of comparison has to be established. Indeed great caution is required with higher frequency spectral components for which $P_i(\delta f)$ are of many orders of magnitude smaller than the maximal spectral component. For these $P_i(\delta f)$ the noise induced by floating point computer arithmetics and the aliasing problem [13] can screen the scaling structure, thus giving rise to completely incorrect evaluation of the multifractal behaviour of the spectrum (4).

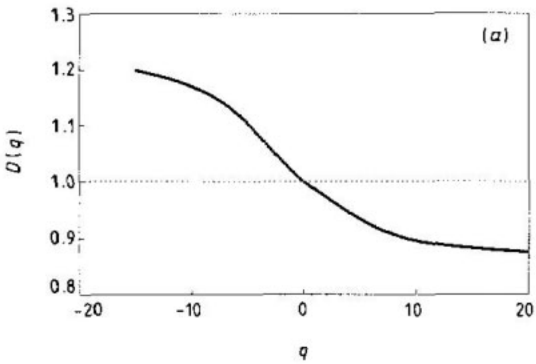
For this reason the criterion adopted for evaluating $\Gamma_q(\delta f)$ was to compute it up to a frequency $f^*(\beta)$ given by:

$$\int_{[0,f^*(\beta)]} \mathrm{d}\mu(f)=\beta \qquad 0<\beta\leqslant 1 \tag{6}$$

In this way the evaluation of multifractal scaling is homogeneously parametrized with respect to the normalized power content β.

Therefore, instead of $D(q)$, $f(\alpha)$ we prefer to write $D(q,\beta)$, $f(\alpha,\beta)$. It is to be noted that computer experiments on the considered systems have shown that the multifractal spectrum is independent of β in the limit $\beta\to 1$.

Figures 4 and 5(a), (b) present the $D(q,\beta)$- and $f(\alpha,\beta)$-spectra of the Lorenz and Rossler systems. As can be seen from these figures, all the components of the same dynamics have similar multifractal spectra. The cusps observed in figure 5(a), (b) are due to experimental noise for high values of $|q|$. The small quantitative differences are due to the experimentally delicate compromise in the value of β. In practice β has to be as close as possible to one in order to cover the overall frequency structure, but not too big in order to avoid the noisy high frequency $P_i(\delta f)$, as previously discussed. Indeed, the increasing behaviour of $D(q,\beta)$-Rossler spectra for q positive is due to experimental errors when $D(q,\beta)$ is close to zero (see figure 4(b)). On the basis of these results it can be stated that the frequency scaling structure is an intrinsic feature of the dynamics and is independent of the choice of the component. This is of course



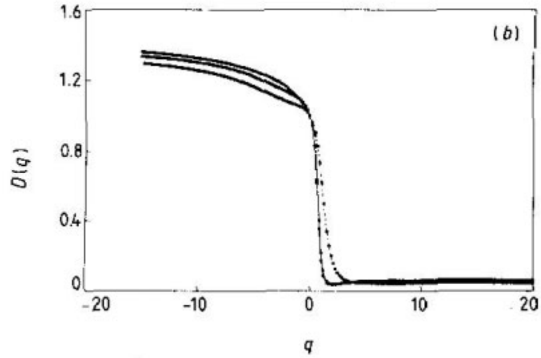
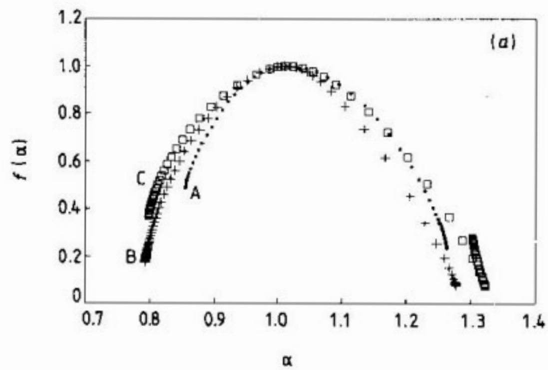


Figure 4. Multifractal $D(q, \beta)$-spectra for Lorenz and Rossler system. Figure 4(*a*) shows the $D(q, \beta)$spectrum of the x-compoent of Lorenz system, with $\beta = 0.99$. Figure 4(*b*) shows the $D(q, \beta)$-spectrum for the three components of Rossler system with $\beta = 0.99999$.

interesting in time series analysis. It should be said that the only significant discrepancy for different components of the same dynamics has been observed for the values of generalized dimensions of Rossler system in the neighbourhood of $q = 1$. This discrepancy is evident in figure 4(*b*).

Comparison of the two systems brings out a clear distinction in the frequency domain between 'fully developed chaos' (Lorenz system) and 'oscillatory chaos' (Rossler system). The distinction between oscillatory (periodic) chaos and fully developed chaos has been introduced by several authors in connection with the qualitative behaviour of power spectra. Thomas and Grossman [14–15] studied the correlation function and the power spectrum of the discrete quadratic map in periodic chaos regime, obtained for the bifurcative structure of the map. Roughly speaking periodic chaos is characterized by a dynamical evolution that still maintain traces of periodic structure while developed chaos has no traces of periodic structure anymore [16]. Ideda and Akimoto explained the route periodic chaos → developed chaos in connection with optical turbulence [16] as a breaking of dominance of periodic isomeric structures. Similar behaviour has been observed by Vidal in analysing the bifurcation route of the Belousov–Zhabotinsky reaction [17].

In terms of multifractal analysis a more quantitative distinction between oscillatory chaos and fully developed chaos can be stated as follows: the former is characterized by a smooth $D(q, \beta)$-spectrum; the latter by a sharp transition for negative and positive



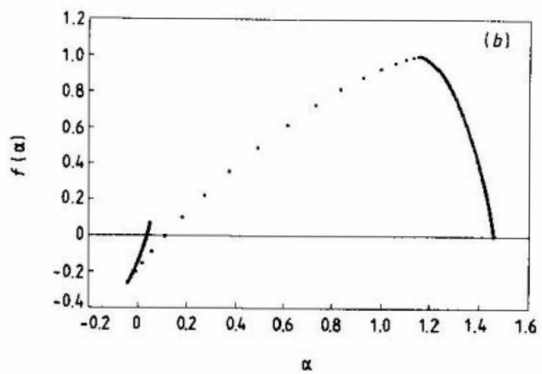


Figure 5. Multifractal $f(\alpha, \beta)$-spectra for Lorenz and Rossler system. Figure 5(*a*) shows the $f(\alpha, \beta)$-spectra of the three components A x-comp, B y-comp, C z-comp, $\beta = 0.99$. Figure 5(*b*) shows the $f(\alpha, \beta)$-spectrum for Rossler z-component with $\beta = 0.99999$.

values of q. In this sense the difference $K(\beta) = D(-\infty, \beta) - D(\infty, \beta)$ represents a useful short-cut parameter for distinguishing periodic and developed chaos. If $K(\beta)$ is small the chaotic frequency structure is homogeneously distributed revealing the absence of dominant periodic peaks. In particular $K(\beta)$ is equal to zero for white noise.

The frequency behaviour of oscillatory chaos can be intuitively understood if the frequency spectrum is regarded as the superimposing of a regular multifractal spectrum and of periodic dominant components. For negative values of q, the fractal structure prevails; for positive q values the strong periodicity of the signal causes the $D(q, \beta)$ values to collapse rapidly to zero. Our computer experiments seem to suggest the hypothesis of a phase-transition [18–20], in the $f(\alpha, \beta)$ frequency domain spectra in the neighbourhood of $q = 0$, due to competition in the multifractal structure between these two different types of dynamical behaviour, figure 5. The fact that our computer simulation experiments seem to lead to a phase-transition at $q = 0$ could be related to the violation of the decorrelation assumption [14]: if we decompose the power spectrum into a periodic part and a chaotic part, these two contributions are correlated. In absence of this correlation one may expect a transition in the $D(q, \beta)$spectrum at $q = 1$.

To sum up, we have shown that frequency domain analysis using a multifractal approach can give interesting results in understanding the structure of chaotic signals. Much work still remains to be done particularly with regard to elucidating the relationship between multifractal analysis of frequency spectra and the corresponding analysis of the attractor in phase space. Furthermore the phase-transition hypothesis in oscillatory chaos needs further investigation to confirm its validity. Finally it should be emphasized that the method discussed in this work is directly applicable to time series analysis.

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